

# EXTREME POINTS METHOD AND UNIVALENT HARMONIC MAPPINGS

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**ABSTRACT.** We consider the class of all sense-preserving complex-valued harmonic mappings  $f = h + \bar{g}$  defined on the unit disk  $\mathbb{D}$  with the normalization  $h(0) = h'(0) - 1 = 0$  and  $g(0) = g'(0) = 0$  with the second complex dilatation  $\omega : \mathbb{D} \rightarrow \mathbb{D}$ ,  $g'(z) = \omega(z)h'(z)$ . In this paper, the authors determine sufficient conditions on  $h$  and  $\omega$  that would imply the univalence of harmonic mappings  $f = h + \bar{g}$  on  $\mathbb{D}$ .

## 1. PRELIMINARIES AND MAIN RESULTS

Denote by  $\mathcal{A}$  the class of all functions  $h$  analytic in the unit disk  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  with the normalization  $h(0) = 0 = h'(0) - 1$ . We let  $\mathcal{S}$  denote the subset of functions from  $\mathcal{A}$  that are univalent in  $\mathbb{D}$ . A locally univalent function  $h \in \mathcal{A}$  is in  $\mathcal{S}^*(\alpha)$  if and only if  $\operatorname{Re}(zh'(z)/h(z)) > \alpha$  for  $z \in \mathbb{D}$  where  $\alpha < 1$ . A locally univalent function  $h \in \mathcal{A}$  is said to belong to  $\mathcal{K}(\alpha)$  if and only if  $zh' \in \mathcal{S}^*(\alpha)$ . Functions in  $\mathcal{S}^*(0)$  and  $\mathcal{K}(0)$  are referred to as the normalized starlike (with respect to  $h(0) = 0$ ) and convex functions in  $\mathbb{D}$ , respectively. Finally,  $h \in \mathcal{A}$  is called close-to-convex if there exists a  $g \in \mathcal{S}^*(0)$  such that  $\operatorname{Re}(e^{i\gamma}zh'(z)/g(z)) > 0$  for  $z \in \mathbb{D}$  and for some  $|\gamma| < \pi/2$ . It is known that every close-to-convex function is univalent (see [10, 21]).

We remind the reader that a univalent analytic or harmonic function  $f$  on  $\mathbb{D}$  is close-to-convex if  $f(\mathbb{D})$  is close-to-convex, i.e. its complement in  $\mathbb{C}$  is the union of closed half lines with pairwise disjoint interiors. We direct the reader to [8, 11] and expository notes [22] for several basic knowledge on planar univalent harmonic mappings and methods of constructing them.

Our first problem concerns the class  $\mathcal{K}(\beta)$  of functions  $h \in \mathcal{S}$  such that

$$(1) \quad \operatorname{Re} \left( 1 + \frac{zh''(z)}{h'(z)} \right) > \beta \quad \text{for } z \in \mathbb{D},$$

for some  $\beta \in [-1/2, 1)$ . For convenience, we let  $\mathcal{K}(-1/2) = \mathcal{F}$ . In particular, functions in  $\mathcal{F}$  are known to be close-to-convex but are not necessarily starlike in  $\mathbb{D}$ . For  $\beta \geq 0$ , functions in  $\mathcal{K}(\beta)$  are known to be convex in  $\mathbb{D}$ .

There are two important sufficient conditions for close-to-convexity of harmonic mappings due to Clunie and Sheil-Small [8]. We now recall them here for a ready reference.

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**Lemma A.** *If a harmonic mapping  $f = h + \bar{g}$  satisfies the condition  $|g'(0)| < |h'(0)|$  and that the analytic function  $h + \epsilon g$  is close-to-convex for each  $\epsilon$  ( $|\epsilon| = 1$ ), then  $f$  is also close-to-convex.*

**Lemma B.** *Let  $h$  be analytic and convex in  $\mathbb{D}$ . If  $g$  and  $\omega$  are analytic in  $\mathbb{D}$  such that  $|\omega(z)| < 1$  and  $g'(z) = \omega(z)h'(z)$  for  $z \in \mathbb{D}$ , then every harmonic mapping of the form  $f = h + \bar{g}$  is close-to-convex and univalent in  $\mathbb{D}$ .*

In our proof, we observe that Lemma B is an immediate consequence of Lemma A (see also the proof of the case  $\beta = 0$  of Theorem C(2)).

The function  $\omega : \mathbb{D} \rightarrow \mathbb{D}$  satisfying the relation  $g' = \omega h'$  is called the second complex dilatation of the sense-preserving harmonic mapping  $f = h + \bar{g}$  in  $\mathbb{D}$ . In our discussion, it is convenient to consider harmonic functions  $f = h + \bar{g}$  in  $\mathbb{D}$  with the standard normalization, namely,  $h(0) = 0 = h'(0) - 1$  and  $g(0) = 0$ , and the family of normalized harmonic convex (resp. close-to-convex and starlike) mappings, i.e. sense-preserving univalent harmonic functions that have a convex (resp. close-to-convex and starlike) range (see [8, 11, 22]).

As an application of Lemma A and Kaplan's characterization of close-to-convex functions, the following results were obtained in [6, 7] (see also Bharanedhar and Ponnusamy [4]).

**Theorem C.** *Let  $f = h + \bar{g}$  be a harmonic mapping in  $\mathbb{D}$  such that  $g'(z) = \omega(z)h'(z)$  in  $\mathbb{D}$  for some  $\omega : \mathbb{D} \rightarrow \mathbb{D}$ . Then  $f$  is close-to-convex in  $\mathbb{D}$  if one of the following conditions is satisfied:*

- (1)  $h \in \mathcal{K}(-1/2)$  and  $\omega(z) = e^{i\theta}z$  in  $\mathbb{D}$
- (2)  $h \in \mathcal{K}(\beta)$  for some  $\beta \in (-1/2, 0]$  and  $|\omega(z)| < \cos(\beta\pi)$  for  $z \in \mathbb{D}$ .

Originally, Theorem C(1) was a conjecture of Mocanu [15] and was settled by Bshouty and Lyzzaik [6] (see also [4]) whereas Theorem C(2) extends Lemma B (see [7, Theorem 4] and [24]). We remark that the case  $\beta = 0$  of Theorem C(2) is equivalent to Lemma B.

In Section 2, using extreme points method, we present an elegant proof of Theorem C and several other new results. Second consequence of our method gives for example the following.

**Theorem 1.** *Let  $h \in \mathcal{F}$ . Then for  $\beta > 0$  and  $r \in (0, 1)$  one has*

$$I_\beta(r, f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta}{|h'(re^{i\theta})|^{2\beta}} \leq \frac{2^{6\beta}}{\pi} \mathbf{B}\left(\frac{6\beta+1}{2}, \frac{1}{2}\right),$$

where  $\mathbf{B}(\cdot, \cdot)$  denotes the Euler-beta function. The inequality is sharp.

We present two different proofs of Theorem 1. One of the proofs relies on the method of extreme points (see for example [12]) while the other relies on the subordination relation. The estimates of  $I_1$  has received special attention in the field of planar fluid mechanics, where these functionals are participating in isoperimetric problems for moving phase domains, eg. [27] and [28].

In order to present the third consequence of our approach, we recall the following result which is a partial extension of the classical result of Alexander's theorem from conformal mappings to univalent harmonic mappings.

**Theorem D.** ([11, p.108, Lemma]) *Let  $f = h + \bar{g}$  be a sense-preserving harmonic starlike mapping in  $\mathbb{D}$ . If  $H$  and  $G$  are the analytic functions defined by the relations*

$$(2) \quad zH'(z) = h(z), \quad zG'(z) = -g(z), \quad H(0) = G(0) = 0,$$

*then  $F = H + \bar{G}$  is a convex mapping in  $\mathbb{D}$ .*

A generalization of Theorem D has been obtained by Ponnusamy and Sairam Kaliraj [24]. However, it is natural to ask what would be the conclusion if the assumption about  $f$  is replaced just by the analytic part  $h$  being starlike in  $\mathbb{D}$ . We remark that the harmonic Koebe function (see [8, 11, 22])  $K$  defined by

$$K(z) = \frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3} + \overline{\left( \frac{\frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3} \right)} \quad \text{for } z \in \mathbb{D},$$

is starlike in  $\mathbb{D}$  whereas its analytic part is not even univalent in  $\mathbb{D}$ . Also, there are harmonic convex function whose analytic part is not necessarily starlike in  $\mathbb{D}$ .

**Theorem 2.** *Let  $f = h + \bar{g}$  be a sense-preserving harmonic mapping in  $\mathbb{D}$ , where  $h \in \mathcal{S}^*$  and  $g(0) = 0$ . If  $H$  and  $G$  are the analytic functions defined by the relations (2), then for each  $|\lambda| \leq 1$ , the harmonic function  $F_\lambda = H + \lambda\bar{G}$  is sense-preserving and close-to-convex mapping in  $\mathbb{D}$ . In particular,  $F = H + \bar{G}$  is a close-to-convex mapping in  $\mathbb{D}$ .*

We now state our next result whose proof follows similarly. So we omit its detail.

**Theorem 3.** *Let  $f = h + \bar{g}$  be a harmonic mapping in  $\mathbb{D}$ , where  $h \in \mathcal{S}^*(\beta)$  for some  $\beta \in (-1/2, 0]$ ,  $g(0) = 0$  and  $g'(z) = \omega(z)h'(z)$  in  $\mathbb{D}$  for some  $\omega : \mathbb{D} \rightarrow \mathbb{D}$  satisfying the condition  $|\omega(z)| < \cos(\beta\pi)$  for  $z \in \mathbb{D}$ . If  $H$  and  $G$  are the analytic functions defined by the relations (2), then for each  $|\lambda| = 1$ , the harmonic function  $F_\lambda = H + \lambda\bar{G}$  is sense-preserving and close-to-convex mapping in  $\mathbb{D}$ .*

We remark that functions in  $\mathcal{S}^*(\beta)$  are not necessarily univalent in  $\mathbb{D}$  if  $\beta < 0$ . At the end of the article, Bshouty and Lyzzaik [6] expressed their interest in determining sufficient condition on  $h$  so that  $g'(z) = e^{i\theta}zh'(z)$  implies that  $f = h + \bar{g}$  is univalent in  $\mathbb{D}$ . Several of the remaining results of this article motivate their desire by choosing  $h$  appropriately. Proof of Theorem 2 will be given in Section 2.

For  $\alpha \in (1, 2)$ , let  $CO_H(\alpha)$  denote the class of all harmonic mappings  $f = h + \bar{g}$  defined on  $\mathbb{D}$ , where  $g'(z) = \omega(z)h'(z)$  with  $|\omega(z)| < 1$  for  $z \in \mathbb{D}$  and  $h \in CO(\alpha)$ , the class of all concave univalent functions (see Section 2 for the precise definition). The class  $CO(\alpha)$  has been extensively studied in the recent years and for a detailed discussion about concave functions, we refer to [2, 3, 5, 9] and the references therein. We now state our next result.

**Theorem 4.** *For  $\alpha \in (1, 2)$ , let  $f = h + \bar{g} \in CO_H(\alpha)$ . If the dilatation  $\omega$  satisfies the conditions  $|\omega(z)| < \sin(\frac{2-\alpha}{2}\pi)$  for  $z \in \mathbb{D}$ , then  $f$  is close-to-convex (univalent) in  $\mathbb{D}$ .*

A simple consequence of Theorem 4 gives

**Corollary 1.** *For  $\alpha \in (1, 2)$ , each harmonic mapping  $f \in CO_H(\alpha)$  with the dilatation  $\omega(z) = (\sin(\frac{2-\alpha}{2})\pi) e^{i\theta} z$  for  $z \in \mathbb{D}$  is close-to-convex (univalent) in  $\mathbb{D}$ .*

We conjecture that Corollary 1 is sharp in the sense that the number  $\sin(\frac{2-\alpha}{2})\pi$  cannot be replaced by a larger one for a given  $\alpha \in (1, 2)$ .

A function  $h$  analytic and locally univalent in  $\mathbb{D}$  is said to have *boundary rotation* bounded by  $K\pi$ ,  $K \geq 2$ , if for  $0 < r < 1$

$$(3) \quad \int_0^{2\pi} \left| \operatorname{Re} \left( 1 + \frac{re^{i\theta} h''(re^{i\theta})}{h'(re^{i\theta})} \right) \right| d\theta \leq K\pi.$$

Let  $\mathcal{V}_K$  be the class of all analytic functions  $h$  in  $\mathbb{D}$  (with the normalization  $h(0) = 0 = h'(0) - 1$ ) having boundary rotation bounded by  $K\pi$ . The reader is referred to Paatero [17] (see also [10, 13] and Section 2 for additional information about the class  $\mathcal{V}_K$ ), where the study of these classes was initiated, for the geometric significance.

**Theorem 5.** *Let  $h \in \mathcal{V}_K$  with  $2 \leq K \leq 4 - \delta$  for a fixed  $\delta \in [0, 2]$ , and the dilatation satisfies the condition  $|\omega(z)| < \sin(\frac{\delta\pi}{4})$  for  $z \in \mathbb{D}$ . Then harmonic mapping  $f = h + \bar{g}$  is close-to-convex and univalent in  $\mathbb{D}$ .*

As an immediate corollary to this result, we have

**Corollary 2.** *Let  $f = h + \bar{g}$  be a harmonic mapping in  $\mathbb{D}$  such that  $h \in \mathcal{V}_K$  with  $2 \leq K \leq 4 - \delta$  for a fixed  $\delta \in [0, 2]$ , and that  $g'(z) = e^{i\theta} \sin(\frac{\delta\pi}{4}) z h'(z)$  for  $z \in \mathbb{D}$ . Then  $f$  is close-to-convex and univalent in  $\mathbb{D}$ .*

We conjecture that Corollary 2 is sharp in the sense that the number  $\sin(\frac{\delta\pi}{4})$  cannot be replaced by a larger one for a given  $\delta < 2$ .

Images of  $\mathbb{D}$  under the close-to-convex mappings  $f_{K,\delta}(z) = h(z) + \overline{g(z)}$  for certain values of  $\delta$  and  $K$  with  $2 \leq K \leq 4 - \delta$ , where

$$h(z) = \frac{1}{K} \left[ \left( \frac{1+z}{1-z} \right)^{K/2} - 1 \right] \quad \text{and} \quad g'(z) = \sin \left( \frac{\delta\pi}{4} \right) z h'(z) \quad \text{for } z \in \mathbb{D},$$

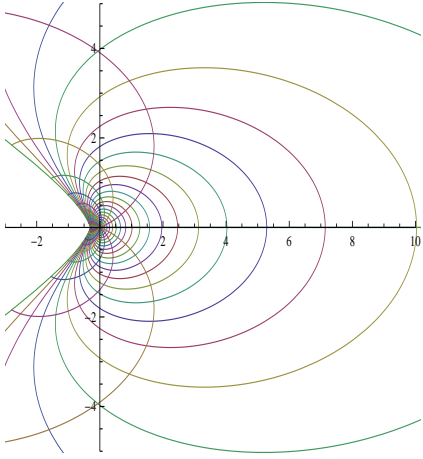
are drawn in Figure 1(a)–(h) using mathematica as plots of the images of equally spaced radial segments and concentric circles of the unit disk.

Finally, we consider the class  $\mathcal{G}$  of functions  $h \in \mathcal{A}$  such that

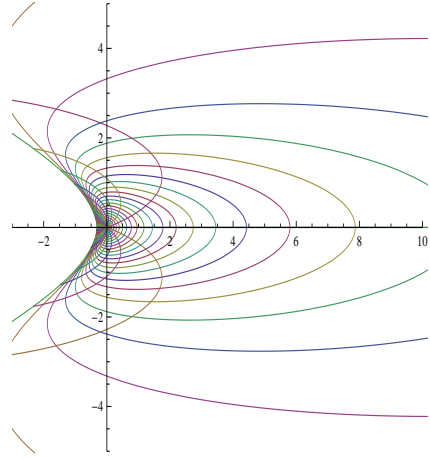
$$(4) \quad \operatorname{Re} \left( 1 + \frac{zh''(z)}{h'(z)} \right) < \frac{3}{2} \quad \text{for } z \in \mathbb{D}.$$

Functions in  $\mathcal{G}$  are known to be starlike in  $\mathbb{D}$ . This class has been discussed recently, see for example [16] and the references therein. For this class we prove the following general result.

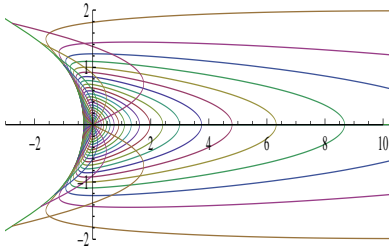
**Theorem 6.** *Suppose that  $h \in \mathcal{G}$  and satisfies the condition  $g'(z) = \omega(z)h'(z)$  in  $\mathbb{D}$ , where  $\omega : \mathbb{D} \rightarrow \mathbb{D}$  is analytic,  $\omega(0) = 0$  and  $W(z) = z(1 + \omega(z))$  is starlike in  $\mathbb{D}$ . Then the harmonic mapping  $f = h + \bar{g}$  is close-to-convex and univalent in  $\mathbb{D}$ .*



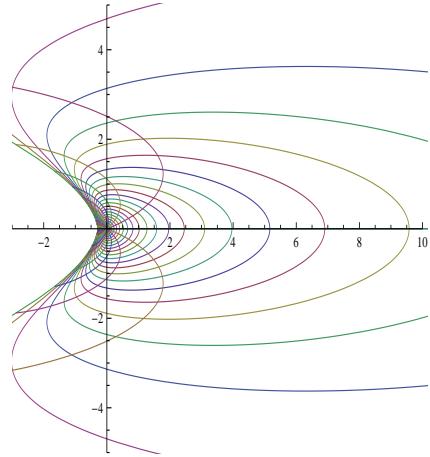
(a)  $\delta = 0.5, K = 3$



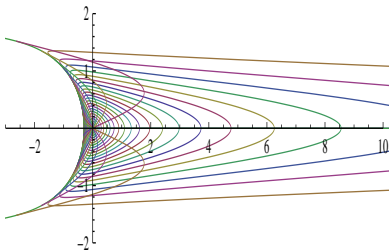
(b)  $\delta = 1, K = 2.5$



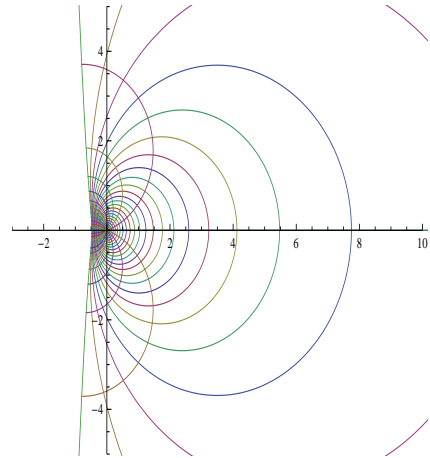
(c)  $\delta = 1.5, K = 2.1$



(d)  $\delta = 1, K = 2.75$



(e)  $\delta = 1.9, K = 2.05$



(f)  $\delta = 0.1, K = 2.05$

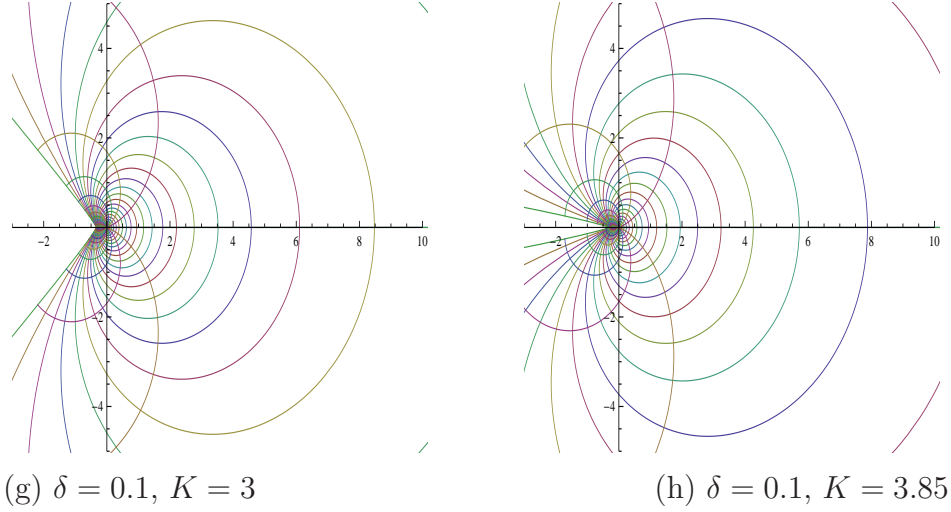


FIGURE 1. The images of unit disk  $\mathbb{D}$  under  $f_{K,\delta}(z) = h(z) + \overline{g(z)}$  for certain values of  $\delta$  and  $K$

**Corollary 3.** *Let  $h \in \mathcal{G}$  and  $g$  be analytic in  $\mathbb{D}$  such that  $g'(z) = \lambda z^n h'(z)$  for some  $n \in \mathbb{N}$  and  $0 < |\lambda| \leq 1/(n+1)$ . Then the harmonic mapping  $f = h + \overline{g}$  is close-to-convex and univalent in  $\mathbb{D}$ .*

*Proof.* Set  $\omega(z) = \lambda z^n$  for  $z \in \mathbb{D}$ . Then  $W(z) = z(1 + \omega(z)) = z + \lambda z^{n+1}$  is starlike in  $\mathbb{D}$  if and only if  $|\lambda| \leq 1/(n+1)$ . Indeed we have

$$|W'(z) - 1| = (n+1)|\lambda| |z|^n < 1 \quad \text{for } z \in \mathbb{D}$$

and hence,  $W$  is univalent in  $\mathbb{D}$ . Further, since  $0 < |\lambda| \leq 1/(n+1)$ , we see that

$$\left| \frac{zW'(z)}{W(z)} - 1 \right| = \left| \frac{n\lambda z^n}{1 + \lambda z^n} \right| < \frac{n|\lambda|}{1 - |\lambda|} \leq 1 \quad \text{for } z \in \mathbb{D}$$

which implies that the function  $W$  is starlike in  $\mathbb{D}$ . The desired conclusion follows from Theorem 6.  $\square$

**Example 5.** According to Corollary 3, it follows that if  $h \in \mathcal{G}$  and  $g$  is analytic in  $\mathbb{D}$  such that  $g'(z) = \lambda z h'(z)$  for some  $\lambda$  with  $|\lambda| \leq 1/2$ , then the harmonic mapping  $f = h + \overline{g}$  is close-to-convex (univalent) in  $\mathbb{D}$ . For instance, let  $h_1(z) = z - z^2/2$  and  $g_1(z) = \lambda \left( \frac{z^2}{2} - \frac{z^3}{3} \right)$ . Then we see that  $f_1 = h_1 + \overline{g_1}$  is clearly locally univalent in  $\mathbb{D}$  for each  $|\lambda| < 1$ . Also,

$$1 + \frac{zh_1''(z)}{h_1'(z)} = \frac{1-2z}{1-z} \quad \text{for } z \in \mathbb{D}$$

and, since  $w = (1-2z)/(1-z)$  maps  $\mathbb{D}$  onto the half-plane  $\operatorname{Re} w < 3/2$ , by Corollary 3, it follows that  $f_1 = h_1 + \overline{g_1}$  is close-to-convex in  $\mathbb{D}$  for each  $\lambda$  with  $|\lambda| \leq 1/2$ .

We conjecture that Corollary 3 is sharp in the sense that the bound on  $\lambda$  cannot be improved.

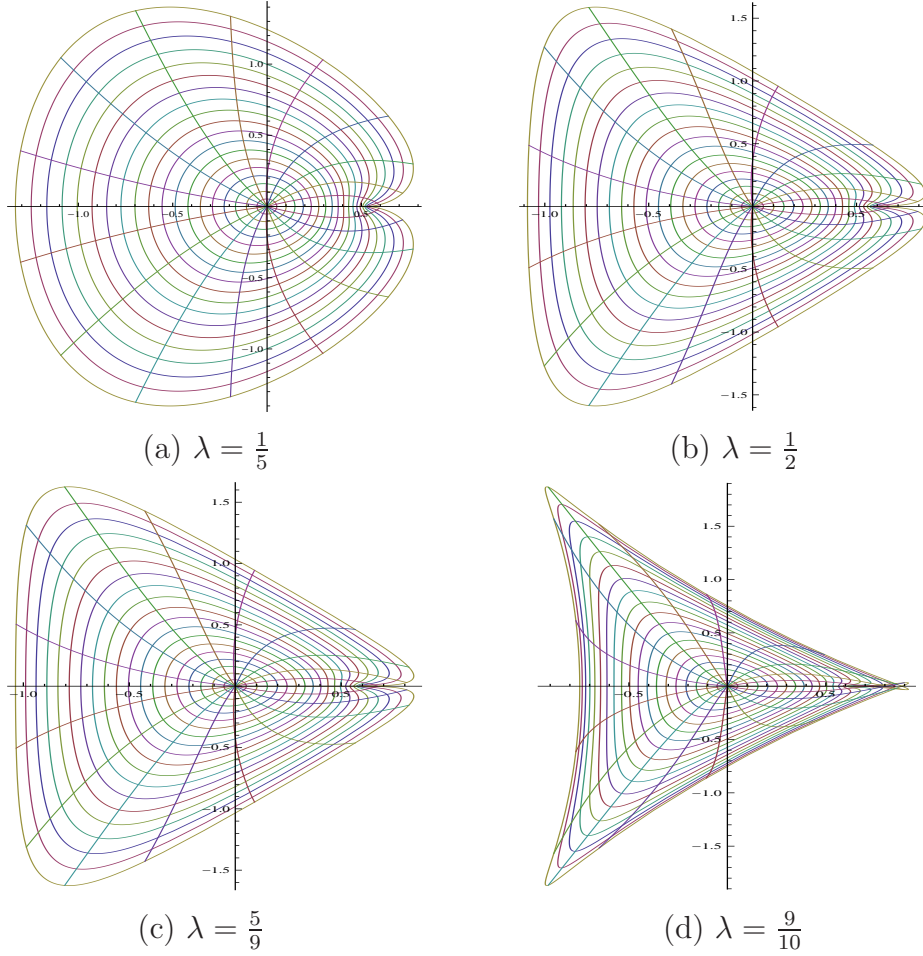


FIGURE 2. Image of  $\mathbb{D}$  under  $f(z) = z - \frac{z^2}{2} + \lambda \left( \frac{z^2}{2} - \frac{z^3}{3} \right)$

The image of unit disk under the function  $f(z) = z - \frac{z^2}{2} + \lambda \left( \frac{z^2}{2} - \frac{z^3}{3} \right)$  for the values of  $\lambda = 1/5, 1/2, 5/9, 9/10$  are drawn in Figure 2(a)–(d) using mathematica as plots of the images of equally spaced radial segments and concentric circles of the unit disk. Closer examination of Figure 2(c)–(d) shows that the functions in these two cases are not univalent in  $\mathbb{D}$ .

## 2. PROOFS OF MAIN THEOREMS

2.1. **The class  $\mathcal{F}$ .** Let  $h \in \mathcal{K}(\beta)$  for some  $-1/2 \leq \beta < 1$ . Then

$$1 + \frac{zh''(z)}{h'(z)} \prec p(z) = \frac{1 + (1 - 2\beta)z}{1 - z} \quad \text{for } z \in \mathbb{D}.$$

Here  $\prec$  denotes the usual subordination (see [10, 21]), and note that  $\operatorname{Re} p(z) > \beta$  in  $\mathbb{D}$ . Thus, by the Herglotz representation for analytic functions with positive real



part in the unit disk, it follows that

$$\frac{zh''(z)}{h'(z)} = 2(1 - \beta) \int_{\partial\mathbb{D}} \frac{\bar{x}z}{1 - \bar{x}z} d\mu(x) \quad \text{for } z \in \mathbb{D},$$

where  $\mu$  is a probability measure on  $\partial\mathbb{D}$  so that  $\int_{\partial\mathbb{D}} d\mu(x) = 1$ . Therefore,

$$\log h'(z) = -2(1 - \beta) \int_{\partial\mathbb{D}} \log(1 - \bar{x}z) d\mu(x) \quad \text{for } z \in \mathbb{D}$$

and thus, we have the sequence of functions  $\{h_n(z)\}$  analytic in  $\mathbb{D}$ ,

$$(6) \quad h'_n(z) = \prod_{k=1}^n (1 - \bar{x}_k z)^{-2(1-\beta)t_k} \quad \text{where } |x_k| = 1, 0 \leq t_k \leq 1, \sum_{k=1}^n t_k = 1,$$

which is dense in the family  $\mathcal{K}(\beta)$ . The representation (6) and Lemma A are two ingredients in the proof of Theorem C and a similar approach helps to prove several other results.

*Proof of Theorem C.* Let  $f = h + \bar{g}$ , where  $h \in \mathcal{K}(\beta)$  for some  $\beta \in [-1/2, 0]$ , and  $g'(z) = \omega(z)h'(z)$  for  $z \in \mathbb{D}$ . Then, it suffices to consider  $h'(z)$  of the form

$$(7) \quad h'(z) = \prod_{k=1}^n (1 - \bar{x}_k z)^{-2(1-\beta)t_k} \quad \text{for } z \in \mathbb{D}.$$

Set

$$S(z) = \frac{z}{\prod_{k=1}^n (1 - \bar{x}_k z)^{2t_k}} \quad \text{for } z \in \mathbb{D}.$$

Then  $S$  is starlike in  $\mathbb{D}$ .

**Case (1):**  $\beta = -1/2$  and  $\omega(z) = e^{i\theta}z$  for  $z \in \mathbb{D}$ .

In this case,  $g'(z) = e^{i\theta}zh'(z)$  and thus, we may rewrite  $h'(z)$  in the form

$$h'(z) = \prod_{k=1}^n (1 - \bar{x}_k z)^{-3t_k} = \left( \prod_{k=1}^n \frac{1}{(1 - \bar{x}_k z)^{t_k}} \right) \frac{S(z)}{z} \quad \text{for } z \in \mathbb{D},$$

where  $|x_k| = 1$ ,  $0 \leq t_k \leq 1$  for  $k = 1, \dots, n$ ,  $\sum_{k=1}^n t_k = 1$ .

Now, for each  $|\epsilon| = 1$  and  $|\lambda| = 1$ , we consider the function

$$A(z) = \frac{z(h'(z) + \epsilon\bar{\lambda}g'(z))}{S(z)} = \frac{1 + \epsilon\bar{\lambda}e^{i\theta}z}{\prod_{k=1}^n (1 - \bar{x}_k z)^{t_k}} = \prod_{k=1}^n (\phi_k(z))^{t_k} \quad \text{for } z \in \mathbb{D},$$

where

$$\phi_k(z) = \frac{1 + \epsilon\bar{\lambda}e^{i\theta}z}{1 - \bar{x}_k z} \quad \text{for } z \in \mathbb{D}.$$

Then the function  $A(z)$  has the property that  $\operatorname{Re}(e^{i\gamma}A(z)) > 0$  in  $\mathbb{D}$  for some  $\gamma$ . In fact for each  $k$ , the function  $\phi_k(z)$  maps the unit disk  $\mathbb{D}$  onto a half plane so that  $\operatorname{Re}(e^{i\theta_k}\phi_k(z)) > 0$  for some  $\theta_k$ . Setting  $\gamma = \sum_{k=1}^n t_k\theta_k$ , we have

$$|\arg(e^{i\gamma}A(z))| = \left| \arg \prod_{k=1}^n (e^{i\theta_k}\phi_k(z))^{t_k} \right| \leq \sum_{k=1}^n t_k |\arg(e^{i\theta_k}\phi_k(z))| < \frac{\pi}{2} \sum_{k=1}^n t_k = \frac{\pi}{2}$$



and hence,  $\operatorname{Re}(e^{i\gamma}A(z)) > 0$  in  $\mathbb{D}$ . It follows that  $F(z) = h(z) + \epsilon\bar{\lambda}g(z)$  is close-to-convex in  $\mathbb{D}$  for each  $|\epsilon| = 1$  and  $|\lambda| = 1$ . Hence, by Lemma A, functions  $h + \lambda\bar{g}$  are close-to-convex in  $\mathbb{D}$ , for each  $|\lambda| = 1$ . In particular,  $f = h + \bar{g}$  is close-to-convex in  $\mathbb{D}$ .

**Case (2):** Let  $\beta = 0$  and  $|\omega(z)| < 1$  for  $z \in \mathbb{D}$ . In this case, using the relation  $g'(z) = \omega(z)h'(z)$  for  $z \in \mathbb{D}$ , it follows easily that the function

$$\frac{z(h'(z) + \epsilon g'(z))}{S(z)} = \frac{zh'(z)(1 + \epsilon\omega(z))}{S(z)} = 1 + \epsilon\omega(z)$$

has positive real part in  $\mathbb{D}$  and hence, by Lemma A, the  $h + \bar{g}$  is close-to-convex in  $\mathbb{D}$ .

Next we assume that  $\beta \in (-1/2, 0)$  and  $|\omega(z)| < c = \cos(\beta\pi)$  for  $z \in \mathbb{D}$ . In this case, using (7) we need to consider the function

$$A(z) = \frac{z(h'(z) + \epsilon\bar{\lambda}g'(z))}{S(z)} = (1 + \epsilon\bar{\lambda}\omega(z)) \prod_{k=1}^n (1 - \bar{x}_k z)^{2\beta t_k}$$

for each  $|\epsilon| = 1$  and  $|\lambda| = 1$ . Again, it suffices to show that  $\operatorname{Re}(e^{i\gamma}A(z)) > 0$  in  $\mathbb{D}$  for some  $\gamma$ . In fact each fractional transformation of the form  $w = \psi_k(z) = 1/(1 - \bar{x}_k z)$  maps the unit disk  $\mathbb{D}$  onto the half plane  $\operatorname{Re} w > 0$  so that

$$\left| \arg(1 - \bar{x}_k z)^{2\beta t_k} \right| \leq \pi|\beta|t_k.$$

and, since  $|\arg(1 + \epsilon\bar{\lambda}\omega(z))| < \arcsin c$  for  $z \in \mathbb{D}$ , it follows that

$$|\arg A(z)| < \arcsin c + \pi|\beta| \sum_{k=1}^n t_k = \arcsin c + \pi|\beta| = \pi/2.$$

It follows that  $F(z) = h(z) + \epsilon\bar{\lambda}g(z)$  is close-to-convex in  $\mathbb{D}$  for each  $|\epsilon| = 1$  and  $|\lambda| = 1$  and the desired conclusion follows from Lemma A.  $\square$

*Proof of Theorem 1.* We can assume that  $h$  belongs to the dense set of  $\mathcal{K}(-1/2)$ , so in view of (6) we have

$$h'(z) = \frac{1}{\prod_{k=1}^n (1 - e^{-i\theta_k} z)^{3t_k}} \quad \text{for } z \in \mathbb{D},$$

where  $\theta_k \in [0, 2\pi]$ ,  $0 \leq t_k \leq 1$  and  $\sum_{k=1}^n t_k = 1$ . Using the last relation we find that

$$\begin{aligned}
\frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{|h'(re^{i\theta})|^{2\beta}} &= \frac{1}{2\pi} \int_0^{2\pi} \prod_{k=1}^n |1 - re^{i(\theta-\theta_k)}|^{6\beta t_k} d\theta \\
&\leq \frac{1}{2\pi} \int_0^{2\pi} \sum_{k=1}^n t_k |1 - re^{i(\theta-\theta_k)}|^{6\beta} d\theta \\
&= \frac{1}{2\pi} \sum_{k=1}^n t_k \int_0^{2\pi} |1 - re^{i(\theta-\theta_k)}|^{6\beta} d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} |1 - re^{i\theta}|^{6\beta} d\theta \\
&\leq 1 + \sum_{k=1}^{\infty} \left| \binom{3\beta}{k} \right|^2 \\
&= \frac{1}{2\pi} \int_0^{2\pi} |1 - e^{i\theta}|^{6\beta} d\theta \\
&= \frac{2^{3\beta}}{2\pi} \int_0^{2\pi} (1 - \cos \theta)^{3\beta} d\theta \\
&= \frac{2^{6\beta}}{\pi} \int_0^{\pi} \sin^{6\beta}(\theta) d\theta \\
&= \frac{2^{6\beta+1}}{\pi} \int_0^{\pi/2} \sin^{6\beta}(\theta) d\theta \\
&= \frac{2^{6\beta}}{\pi} \mathbf{B}\left(\frac{6\beta+1}{2}, \frac{1}{2}\right).
\end{aligned}$$

The desired conclusion follows. For the proof of sharpness part, we consider the function  $h_0$  defined by

$$(8) \quad h_0(z) = \frac{z - z^2/2}{(1 - z)^2} = \frac{1}{2} \left( \frac{z}{1 - z} + \frac{z}{(1 - z)^2} \right).$$

The function  $h_0$  and its rotations belong to  $\mathcal{F}$ . A computation shows that  $h'_0(z) = (1 - z)^{-3}$  and the rest of the sharpness part follows easily.  $\square$

*Remark.* As an alternate approach to the proof of Theorem 1, we may begin with  $h \in \mathcal{F}$ . Then one has (see for instance [23])  $h'(z) \prec (1 - z)^{-3}$  and, since  $h'(z) \neq 0$  in  $\mathbb{D}$ , it follows that

$$\frac{1}{h'(z)} \prec (1 - z)^3, \quad z \in \mathbb{D}.$$

Thus (as in the proof of Theorem 1 in [25]) we see that

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{|h'(re^{i\theta})|^{2\beta}} \leq \frac{1}{2\pi} \int_0^{2\pi} |1 - re^{i\theta}|^{6\beta} d\theta$$

and the rest of the proof is as above. The desired result follows.

*Proof of Theorem 2.* By the hypothesis, there exists an analytic function  $\omega : \mathbb{D} \rightarrow \mathbb{D}$  such that  $g'(z) = \omega(z)h'(z)$ . As  $\omega(0) = g'(0) \in \mathbb{D}$ , it follows that  $|G'(0)| < |H'(0)| = 1$ . Moreover, since  $h$  is starlike in  $\mathbb{D}$ ,  $H$  is convex. Thus, according to Theorem C(2) with  $\beta = 0$ , it suffices to show that  $F_\lambda$  is sense-preserving in  $\mathbb{D}$ . Indeed, since  $|g'(z)/h'(z)| < 1$  in  $\mathbb{D}$ , we obtain from a well-known result of Robinson [26, p. 30] (see also [14, Corollary 3.1]) that

$$|g(z)/h(z)| = |G'(z)/H'(z)| < 1 \quad \text{in } \mathbb{D}$$

(and at the origin this is treated as the obvious limiting case). Thus,  $F_\lambda = H + \lambda \overline{G}$  is sense-preserving and harmonic in  $\mathbb{D}$ . The desired conclusion follows from Theorem C(2).  $\square$

**2.2. The class  $CO(\alpha)$  of concave univalent functions.** We now consider normalized functions  $h$  analytic in  $\mathbb{D}$  and map  $\mathbb{D}$  conformally onto a domain whose complement with respect to  $\mathbb{C}$  is convex and that satisfy the normalization  $h(1) = \infty$ . Furthermore, we impose on these functions the condition that the opening angle of  $h(\mathbb{D})$  at  $\infty$  is less than or equal to  $\pi\alpha$ ,  $\alpha \in (1, 2]$ . We will denote the family of such functions by  $CO(\alpha)$  and call it as the class of concave univalent functions. We note that for  $h \in CO(\alpha)$ ,  $\alpha \in (1, 2]$ , the closed set  $\mathbb{C} \setminus h(\mathbb{D})$  is convex and unbounded. Also, we observe that  $CO(2)$  contains the classes  $CO(\alpha)$ ,  $\alpha \in (1, 2]$ .

For the proof of Theorem 4, we need the following result due to Avkhadiev and Wirths [3].

**Theorem E.** ([3, Theorem 1]) *The set of functions  $h \in CO(\alpha)$ , with*

$$(9) \quad h'(z) = \frac{\prod_{k=1}^n (1 - e^{it_k} z)^{\beta_k}}{(1 - z)^{\alpha+1}},$$

where  $0 < t_1 < t_2 < \dots < t_n < 2\pi$ ,  $0 < \beta_k \leq 1$  for  $k = 1, \dots, n$ , with  $\sum_{k=1}^n \beta_k = \alpha - 1$ , is dense in  $CO(\alpha)$ .

*Proof of Theorem 4.* Let  $h \in CO(\alpha)$ . Then, according to Theorem E, it suffices to prove the theorem for  $h$  of the form (9), where  $0 < t_1 < t_2 < \dots < t_n < 2\pi$ ,  $0 < \beta_k \leq 1$  for  $k = 1, \dots, n$  with  $\sum_{k=1}^n \beta_k = \alpha - 1$ .

Now, we set  $F = h + \epsilon g$ , where  $|\epsilon| = 1$ . Then, because  $g'(z) = \omega(z)h'(z)$  in  $\mathbb{D}$ , the above representation on  $h'(z)$  gives that

$$F'(z) = (1 + \epsilon\omega(z))h'(z) = \frac{(1 + \epsilon\omega(z))}{(1 - z)^2} \prod_{k=1}^n (\phi_k(z))^{\beta_k}$$

where

$$\phi_k(z) = \frac{1 - e^{it_k} z}{1 - z}.$$

With  $k(z) = \frac{\bar{c}z}{(1-z)^2}$  with  $|c| = 1$ , it follows that

$$\frac{zF'(z)}{k(z)} = c(1 + \epsilon\omega(z)) \prod_{k=1}^n (\phi_k(z))^{\beta_k}.$$

We observe that each  $(\phi_k(z))^{\beta_k}$  forms a wedge at the origin with angle of measure  $\beta_k\pi/2$  and containing the point 1. Hence the product with  $c$  make angles of total less than  $(\alpha - 1)\pi/2$ . Next, we note by hypothesis that  $|\omega(z)| < c = \sin(\frac{2-\alpha}{2})\pi$  for  $z \in \mathbb{D}$ , and thus, we deduce that  $|\arg(1 + \epsilon\omega(z))| < \arcsin c = (2 - \alpha)\frac{\pi}{2}$ . Thus,

$$\left| \arg \left( c(1 + \epsilon\omega(z)) \prod_{k=1}^n (\phi_k(z))^{\beta_k} \right) \right| < (2 - \alpha)\frac{\pi}{2} + (\alpha - 1)\frac{\pi}{2} = \frac{\pi}{2}.$$

Observe that the existence of an unimodular complex constant  $c$  is guaranteed as in the proof of Theorem C. Therefore,

$$\operatorname{Re} (c(1 - z)^2 F'(z)) = \operatorname{Re} \left( \frac{zF'(z)}{k(z)} \right) > 0 \text{ for } z \in \mathbb{D}$$

and hence, for each  $\epsilon$  with  $|\epsilon| = 1$ , the analytic function  $F = h + \epsilon g$  is close-to-convex in  $\mathbb{D}$ . The desired conclusion follows from Lemma A.  $\square$

**2.3. The class  $\mathcal{V}_K$  of functions of bounded boundary rotation.** For the proof of Theorem 5, we need some preparation. We begin to recall the familiar representation obtained by Paatero [17] for functions  $h \in \mathcal{V}_K$ :

$$(10) \quad h'(z) = \exp \left( - \int_0^{2\pi} \log(1 - ze^{-it}) d\mu(t) \right),$$

where  $\mu(t)$  is a real valued function of bounded variation on  $[0, 2\pi]$  with

$$(11) \quad \int_0^{2\pi} d\mu(t) = 2 \quad \text{and} \quad \int_0^{2\pi} |d\mu(t)| \leq K.$$

It is well-known that  $\mathcal{V}_2$  coincides with the class of normalized convex univalent functions and that for  $2 \leq K \leq 4$ , all members of  $\mathcal{V}_K$  are univalent in  $\mathbb{D}$  (see [17]). However, Pinchuk [19] strengthen this result by showing that for  $2 \leq K \leq 4$ , the classes  $\mathcal{V}_K$  consist of all close-to-convex functions. This fact also follows from our Theorem 5. However, each of the classes  $\mathcal{V}_K$  with  $K > 4$  contains non-univalent functions. An extremal function belonging to this class is

$$(12) \quad g_K(z) = \frac{1}{K} \left[ \left( \frac{1+z}{1-z} \right)^{K/2} - 1 \right].$$

It had been shown by Pinchuk [20, Theorem 6.2] that the image of the unit disk  $\mathbb{D}$  under a  $\mathcal{V}_K$  function contains the disk of radius  $1/K$  centered at the origin, and the

functions of the class  $\mathcal{V}_K$  are continuous in  $\overline{\mathbb{D}}$  with the exception of at most  $[K/2+1]$  points on the unit circle  $\partial\mathbb{D}$ . Moreover, it is known that  $h \in \mathcal{V}_K$  if and only if

$$1 + \frac{zh''(z)}{h'(z)} = \left(\frac{K}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{K}{4} - \frac{1}{2}\right) p_2(z)$$

for some  $p_1, p_2 \in \mathcal{P}$ , where  $\mathcal{P}$  denotes the class of functions  $p$  analytic in  $\mathbb{D}$  such that  $p(0) = 1$  and  $\operatorname{Re} p(z) > 0$  in  $\mathbb{D}$ . Thus, it follows that

$$h'(z) = \exp \left( - \int_{|x|=1} \log(1 - xz) (d\mu_1(x) - d\mu_2(x)) \right)$$

where

$$\int_{|x|=1} (d\mu_1(x) - d\mu_2(x)) = 2, \quad \int_{|x|=1} d\mu_1(x) = \frac{K}{2} + 1, \quad \text{and} \quad \int_{|x|=1} d\mu_2(x) = \frac{K}{2} - 1.$$

Consequently, we easily have the following

**Lemma 1.** *If  $h \in \mathcal{V}_K$ , then there exists a sequence of functions  $\{h_n(z)\}$  analytic in  $\mathbb{D}$  such that*

$$(13) \quad h'_n(z) = \frac{\prod_{k=1}^n (1 - \overline{x}_k z)^{\alpha_k}}{\prod_{k=1}^n (1 - \overline{y}_k z)^{\beta_k}},$$

where  $|x_k| = 1$ ,  $|y_k| = 1$ ,  $0 \leq \alpha_k, \beta_k \leq 1$  with

$$(14) \quad \sum_{k=1}^n \alpha_k = \frac{K}{2} - 1 \quad \text{and} \quad \sum_{k=1}^n \beta_k = \frac{K}{2} + 1,$$

and  $\{h_n\}$  converges uniformly on compact subsets of  $\mathbb{D}$ . That is,  $\{h_n(z)\}$  is dense in the family  $\mathcal{V}_K$ .

*Proof of Theorem 5.* Set  $F = h + \epsilon g$ , where  $|\epsilon| = 1$  and  $h \in \mathcal{V}_K$ . In view of Lemma 1, it suffices to choose  $h \in \mathcal{V}_K$  so that

$$h'(z) = \frac{\prod_{k=1}^n (1 - \overline{x}_k z)^{\alpha_k}}{\prod_{k=1}^n (1 - \overline{y}_k z)^{\beta_k}},$$

where  $|x_k| = 1$ ,  $|y_k| = 1$ ,  $0 \leq \alpha_k, \beta_k \leq 1$  satisfying the conditions (14). It is convenient to rewrite the last expression as

$$h'(z) = \prod_{k=1}^n \left( \frac{1 - \overline{x}_k z}{1 - \overline{y}_k z} \right)^{\alpha_k} \cdot \prod_{k=1}^n (1 - \overline{y}_k z)^{t_k}, \quad t_k = \alpha_k - \beta_k.$$

Observe now that  $\sum_{k=1}^n t_k = 2$  and thus, the function  $S$  defined by

$$S(z) = \frac{\bar{c}z}{\prod_{k=1}^n (1 - \bar{y}_k z)^{t_k}} \quad (|c| = 1)$$

is starlike in  $\mathbb{D}$ . Further,

$$F'(z) = h'(z) + \epsilon g'(z) = (1 + \epsilon \omega(z))h'(z)$$

so that

$$\frac{zF'(z)}{S(z)} = c(1 + \epsilon \omega(z)) \prod_{k=1}^n \left( \frac{1 - \bar{x}_k z}{1 - \bar{y}_k z} \right)^{\alpha_k}$$

where (by the hypothesis)

$$\sum_{k=1}^n \alpha_k = \frac{K}{2} - 1 \leq 1 - \frac{\delta}{2}.$$

Note that  $|\arg(1 + \epsilon \omega(z))| < \pi\delta/4$ . This observation shows that (with a suitably defined  $c$  on  $\partial\mathbb{D}$ )

$$\left| \arg \left( \frac{zF'(z)}{S(z)} \right) \right| < \frac{\pi}{2} \left( \sum_{k=1}^n \alpha_k + \frac{\delta}{2} \right) = \frac{\pi}{2} \left( \frac{K}{2} - 1 + \frac{\delta}{2} \right) \leq \frac{\pi}{2}$$

and thus, the function  $zF'(z)/S(z)$  has positive real part in  $\mathbb{D}$ . It follows that  $F(z) = h(z) + \epsilon g(z)$  is close-to-convex in  $\mathbb{D}$  for each  $|\epsilon| = 1$  and hence, by Lemma A, the harmonic function  $f = h + \bar{g}$  is close-to-convex in  $\mathbb{D}$ .  $\square$

**2.4. The class  $\mathcal{G}$ .** Now, we let  $h \in \mathcal{G}$ . Then (4) holds. Clearly, (4) can be written as

$$1 + \frac{zh''(z)}{h'(z)} \prec p(z) = \frac{1 - 2z}{1 - z} \quad \text{for } z \in \mathbb{D}$$

and thus, by the Herglotz representation for analytic functions with positive real part in the unit disk, it follows easily that

$$\frac{h''(z)}{h'(z)} = - \int_{\partial\mathbb{D}} \frac{\bar{x}}{1 - \bar{x}z} d\mu(x) \quad \text{for } z \in \mathbb{D},$$

where  $\mu$  is a probability measure on  $\partial\mathbb{D}$  so that  $\int_{\partial\mathbb{D}} d\mu(x) = 1$ . This means that

$$h'(z) = \exp \left( \int_{\partial\mathbb{D}} \log(1 - \bar{x}z) d\mu(x) \right) \quad \text{for } z \in \mathbb{D}.$$

Thus, we have a sequence of functions  $\{h_n(z)\}$  analytic in  $\mathbb{D}$  such that

$$(15) \quad h'_n(z) = \prod_{k=1}^n (1 - \bar{x}_k z)^{\alpha_k}$$

where  $|x_k| = 1$ ,  $0 \leq \alpha_k \leq 1$  for  $k = 1, 2, \dots, n$ ,  $\sum_{k=1}^n \alpha_k = 1$ , and  $h_n \rightarrow h$  uniformly on compact subsets of  $\mathbb{D}$ . That is,  $\{h_n(z)\}$  is dense in the family  $\mathcal{G}$ . We observe that functions in  $\mathcal{G}$  are bounded in  $\mathbb{D}$ .

*Proof of Theorem 6.* As in the proofs of previous theorems, we begin to set  $F = h + \epsilon g$ , where  $|\epsilon| = 1$  and  $h \in \mathcal{G}$ . In view of the above discussion and (15), it suffices to prove the theorem for functions  $h$  of the form

$$h'(z) = \prod_{k=1}^n (1 - \bar{x}_k z)^{\alpha_k}$$

where  $|x_k| = 1$ ,  $0 \leq \alpha_k \leq 1$  for  $k = 1, 2, \dots, n$  and  $\sum_{k=1}^n \alpha_k = 1$ . Consequently, there exists a complex number  $c$  with  $|c| = 1$  and such that

$$\frac{czF'(z)}{W(z)} = c \prod_{k=1}^n (1 - \bar{x}_k z)^{\alpha_k}$$

has positive real part for  $z \in \mathbb{D}$ , where  $W$  defined by  $W(z) = z + \epsilon z \omega(z)$  is starlike for each  $|\epsilon| = 1$  (by hypothesis). Thus, the harmonic function  $f = h + \bar{g}$  is close-to-convex in  $\mathbb{D}$  (by Lemma A).  $\square$

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